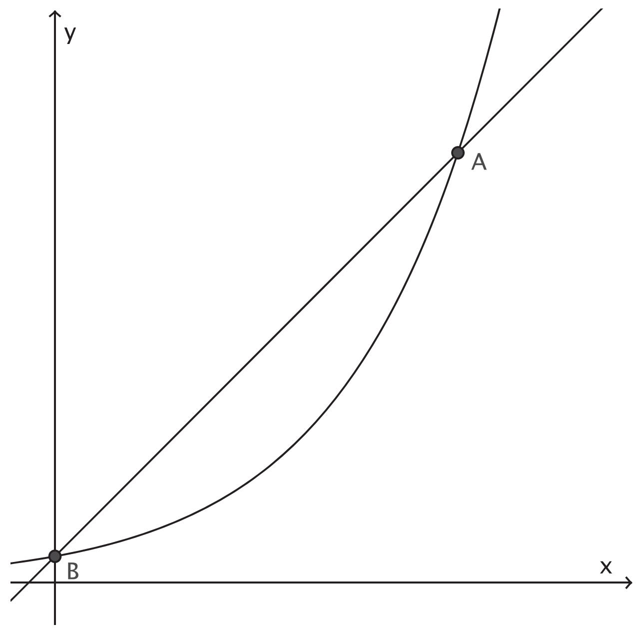
Lesson 24, Part A Linear Approximations of Exponential Functions

1) The following figure shows the graphs of the functions $f(x) = x + 1$ and $g(x) = a^x$. For a particular value of a, the graphs intersect at points A and B, as shown.

   Suppose that the value of a increases. What happens to the locations of points A and B?



Objectives for the lesson

You will understand:

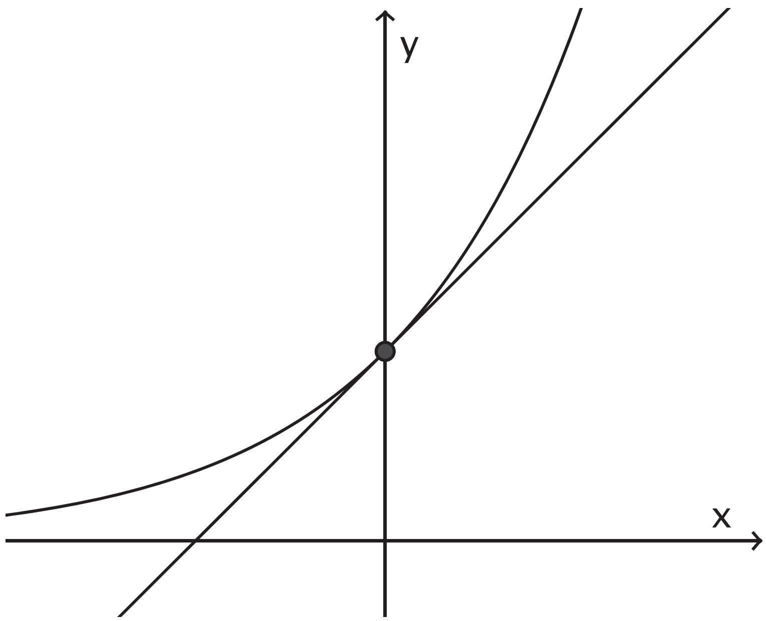
- ☐ Why the number e is a convenient base for exponential functions.
- ☐ That linear functions are good approximations for exponential functions on small intervals.

You will be able to:

- ☐ Find a formula $f(x)$ for an exponential function given a line tangent to the graph of f at $x = 0$.
- ☐ Interpret exponential functions and their linear approximations in the context of a model.

In Preview Assignment 24.A, you learned that a line is **tangent** to a smooth curve at a point P if the line intersects the curve at P without crossing the curve at P. In this activity, we will see how the tangent line can be a useful approximation of a more complicated function.

2) The following graph shows the functions $f(x) = x + 1$ and $g(x) = a^x$, where the value of a is chosen so that the line $y = f(x)$ is tangent to the curve $y = g(x)$ at the point $(0,1)$. Use your graphing calculator or app to estimate this value of a by trying different values of a in the interval $1 \le a \le 4$. Use a viewing window with $-2 \le x \le 2$ and $-1 \le y \le 3$.



3) Now change the viewing window to $-0.001 \le x \le 0.001$ and $0.999 \le y \le 1.001$. Describe what you see. If the graph of f appears to cross the graph of g, adjust your value of a to get a better approximation.

4) Mathematicians use the letter e to denote the exact value for which the graph of $y = x + 1$ is tangent to the curve $y = e^x$ at the point $(0,1)$. Most graphing calculators and apps can produce a numerical approximation to the constant e. Use your graphing calculator or app to give the value of e to five decimal places, and compare this number to your answer to question 2.

In previous lessons, we observed that scaling the input of a function by a positive number stretches (or shrinks) its graph horizontally without changing the vertical intercept. Therefore, if the line $y = f(x)$ is tangent to the curve $y = g(x)$ at the point $(0, f(0))$, then the scaled line $y = f(kx)$ will be tangent to the scaled curve $y = g(kx)$ at the point $(0, f(0))$, for any positive constant k. Use this fact to answer the following question.

5) Based on experimental data, a biologist estimates that, for small values of t, the size of a bacteria colony can be approximated by the linear function $L(t) = 0.034t + 1$, where t is time measured in minutes, and the size of the population is measured as a multiple of its original size. However, the biologist knows that the bacteria population grows exponentially. Find an exponential function in the form $P(t) = e^{kt}$ such that the graph of L is tangent to the graph of P at the point $(0,1)$. Then use the power rule for exponents to write your formula in the form $P(t) = a^t$.

6) Check your answer to question 5 by graphing both L and P on the same pair of axes using a calculator or app. Is $L(t)$ a good approximation for $P(t)$ on the interval $0 \le t \le 10$? How about on the interval $0 \le t \le 90$? Explain.

7) In this activity, we have seen that the linear function L can be used to approximate the exponential function P for some input values.

Part A: How could you estimate the value of $P(10)$ without using a calculator?

Part B: Use the graph of the function P to estimate how long it will take for the bacteria colony to grow to four times its original size. Would the answer be different if you used the linear function L instead of P?