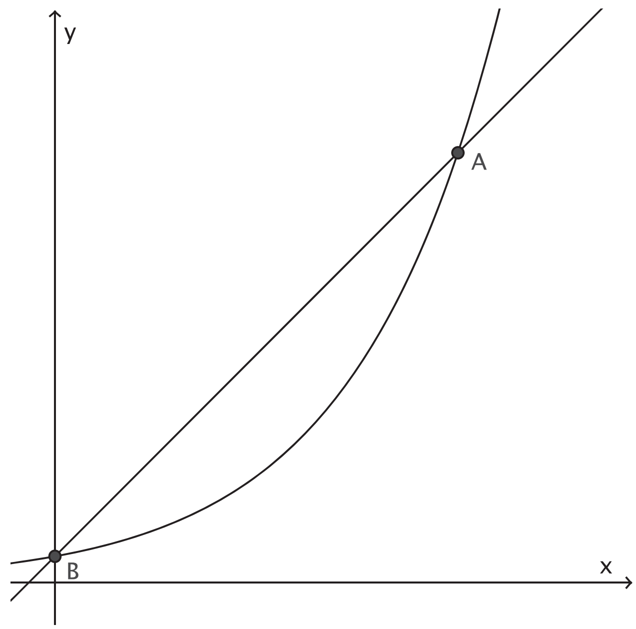
Lesson 24, Part A Linear Approximations of Exponential Functions

1) The following figure shows the graphs of the functions $f(x) = x + 1$ and $g(x) = a^x$. For a particular value of a, the graphs intersect at points A and B, as shown.

   Suppose that the value of a increases. What happens to the locations of points A and B?



Objectives for the lesson

You will understand:

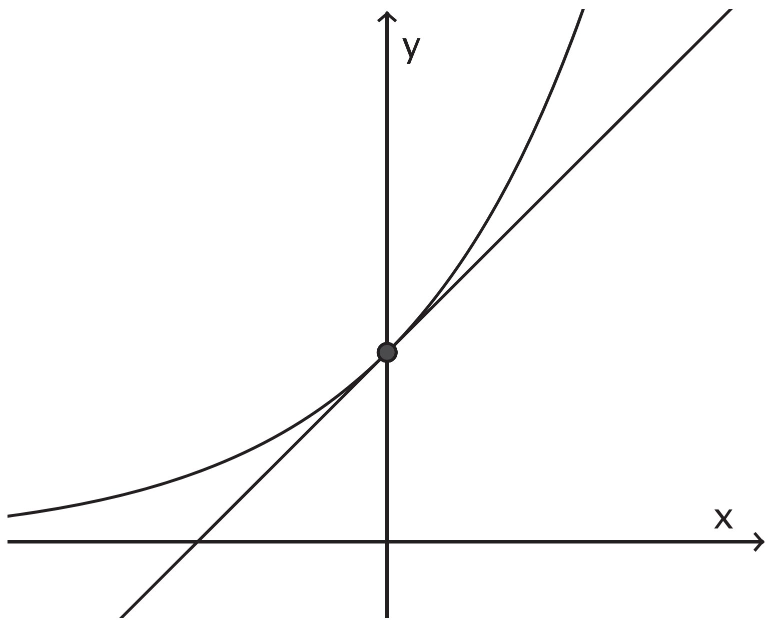
- ☐ Why the number e is a convenient base for exponential functions.
- ☐ That linear functions are good approximations for exponential functions on small intervals.

You will be able to:

- ☐ Find a formula $f(x)$ for an exponential function given a line tangent to the graph of f at $x = 0$.
- ☐ Interpret exponential functions and their linear approximations in the context of a model.

In Preview Assignment 24.A, you learned that a line is **tangent** to a smooth curve at a point P if the line intersects the curve at P without crossing the curve at P. In this activity, we will see how the tangent line can be a useful approximation of a more complicated function.

2) The following graph shows the functions $f(x) = x + 1$ and $g(x) = a^x$, where the value of a is chosen so that the line $y = f(x)$ is tangent to the curve $y = g(x)$ at the point $(0,1)$. Use your graphing calculator or app to estimate this value of a by trying different values of a in the interval $1 \le a \le 4$. Use a viewing window with $-2 \le x \le 2$ and $-1 \le y \le 3$.



3) Now change the viewing window to $-0.001 \le x \le 0.001$ and $0.999 \le y \le 1.001$. Describe what you see. If the graph of f appears to cross the graph of g, adjust your value of a to get a better approximation.

4) Mathematicians use the letter e to denote the exact value for which the graph of $y = x + 1$ is tangent to the curve $y = e^x$ at the point $(0,1)$. Most graphing calculators and apps can produce a numerical approximation to the constant e. Use your graphing calculator or app to give the value of e to five decimal places, and compare this number to your answer to question 2.

In previous lessons, we observed that scaling the input of a function by a positive number stretches (or shrinks) its graph horizontally without changing the vertical intercept. Therefore, if the line $y = f(x)$ is tangent to the curve $y = g(x)$ at the point $(0, f(0))$, then the scaled line $y = f(kx)$ will be tangent to the scaled curve $y = g(kx)$ at the point $(0, f(0))$, for any positive constant k. Use this fact to answer the following question.

5) Based on experimental data, a biologist estimates that, for small values of t, the size of a bacteria colony can be approximated by the linear function $L(t) = 0.034t + 1$, where t is time measured in minutes, and the size of the population is measured as a multiple of its original size. However, the biologist knows that the bacteria population grows exponentially. Find an exponential function in the form $P(t) = e^{kt}$ such that the graph of L is tangent to the graph of P at the point $(0,1)$. Then use the power rule for exponents to write your formula in the form $P(t) = a^t$.

6) Check your answer to question 5 by graphing both L and P on the same pair of axes using a calculator or app. Is $L(t)$ a good approximation for $P(t)$ on the interval $0 \le t \le 10$? How about on the interval $0 \le t \le 90$? Explain.

7) In this activity, we have seen that the linear function L can be used to approximate the exponential function P for some input values.

Part A: How could you estimate the value of $P(10)$ without using a calculator?

Part B: Use the graph of the function P to estimate how long it will take for the bacteria colony to grow to four times its original size. Would the answer be different if you used the linear function L instead of P?

Lesson 24, Part A

Linear Approximations of Exponential Functions

Overview and student objectives

Overview

Students will discover how the constant e arises naturally when approximating an exponential function with a linear function. Students will use technology to confirm that, at $x=0$, the function $L(x)=x+1$ is a linear approximation of $f(x)=e^x$. More general approximations can be found by scaling the input of these functions.

Objectives

Students will understand:

- Why the number e is a convenient base for exponential functions.
- That linear functions are good approximations for exponential functions on small intervals.

Lesson Length: 25 minutes

Prior Lesson: Lesson 23, Part C, "Surge Functions"

Next Lesson: Lesson 24, Part B, "Compound Interest" (25 minutes)

Constructive Perseverance Level: 2

Outcomes: FF3, FF4, FF5, AF1, AF2

Goals: Communication, Reasoning, Technology

Related Foundations Outcomes: A1, A2, A5, A6, A7

Students will be able to:

- Find a formula $f(x)$ for an exponential function given a line tangent to the graph of f at $x=0$.
- Interpret exponential functions and their linear approximations in the context of a model.

Suggested resources and preparation

Materials and technology

- Computer, projector, document camera
- Preview Assignment 24.A
- Student Pages for Lesson 24, Part A
- Practice Assignment 24.A
- A graphing application with sliders, such as the Desmos app.

Prerequisite assumptions

Students should be able to:

- Recognize when a line is tangent to a curve.
- Scale the input of a function.
- Use the power rule for exponents.

Making connections

This lesson:

- Connects back to linear functions, exponential functions, and exponential growth.
- Connects forward to modeling with exponentials and natural logarithms.
- Connects to biology and calculus.

Background context

Preview Assignment 24.A introduced an informal definition of the tangent line to a curve at a point. This definition is also given in this activity.

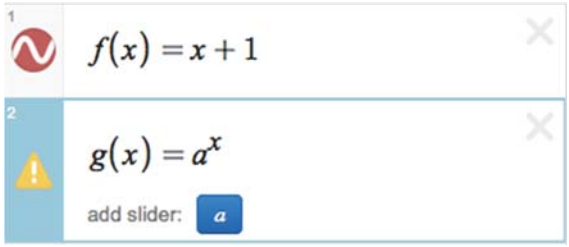
Suggested instructional plan

Frame the lesson

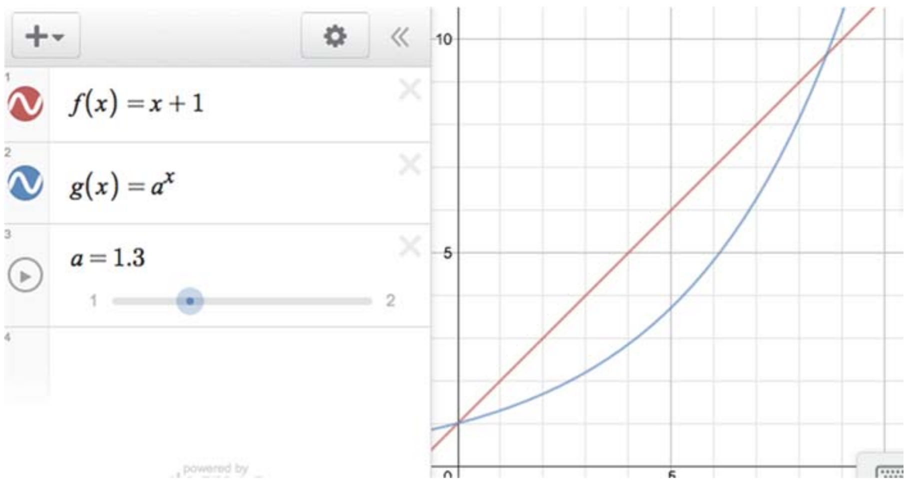
(4 minutes)

Student Page

- Have students discuss question 1 briefly in pairs. Ask for one or two volunteers to share their answers.
- Before verifying any conjectures graphically, try to get students to reason through the algebraic formula. For example, “If the base increases, then the new base raised to a power should be greater than the old base raised to the same power. This should make the graph steeper, so it will intersect the line sooner.
- Demonstrate the answer to question 1 on the computer projector using a graphing app that has sliders. For example, the Desmos app (www.desmos.com/calculator) will prompt the user to add a slider for the variable a when the function $g(x) = a^x$ is entered:



Click on the “add slider” button, and a slider will appear that allows you to change the value of a. Change the endpoints of the slider to 1 and 2, and demonstrate how changing the value of a changes the shape of the exponential graph. You can also press the play button and the app will animate the changing values of a.



- Since many graphing calculators lack the pixel density needed to observe the difference between these two graphs, it is best to use the Desmos app (or a similar app) for these demonstrations.

Framing Statement

- “A line can make a nice simple approximation for a more complicated function. In this activity, we will see how to use tangent lines to approximate exponential functions near the point of tangency.”
- Transition to the lesson activities by briefly discussing the **Objectives for the lesson**.

Lesson activities

(15 minutes)

Student Equipment

- Students will need graphing devices to participate in this lesson. While the lesson can be done with graphing calculators, students might work more efficiently if they can use an app that has sliders, such as the Desmos app.

Optional Mini-Lecture

Questions 2-3

- It is best to have students investigate questions 2-4 using their graphing devices. However, it is possible that students with older graphing calculators may have trouble finishing the lesson in the allotted time. If students don't have access to graphing devices with sliders, instructors can save time by doing questions 2 and 3 as a demonstration using the Desmos app on the projector.

Group Work

Questions 2-4

- Have students work in groups on questions 2-4.
- The accuracy of student answers for questions 2 and 3 will depend on the screen resolution of their graphing devices. Students using older graphing calculators will probably get answers between 2.4 and 3.0 for question 2, while students who use apps will have more accurate answers.
- Students who use calculators or apps without sliders will have to adjust the value of a by editing the formula for the function.

Class Discussion

Question 4

- After most of the groups have completed question 3, interrupt the group time and ask each group for the value of a that they found in question 3. Write these estimates on the board.
- On the graph that you projected for question 1, change the viewing window to $-0.001 \le x \le 0.001$ and $0.999 \le y \le 1.001$. Change the range of the slider to go from 1 to 4. Manipulate the slider to test the different group responses.
- "Notice that for values of x near zero, the graph of the exponential function is almost identical to the graph of the linear function.

Literacy Support

Question 4

- The constant $e \approx 2.71828182846$ is sometimes called "Euler's number." Students have seen formulas using e in previous lessons, so they should remember how to find it on their calculators. Questions 2-4 introduce e as the unique number for which the slope of the tangent line to $y = e^x$ at $x = 0$ is 1. In fact, the slope of the tangent line to $y = e^x$ at *any* point $x = c$ is e^c. There are other ways to define the constant e; the next activity will present another context in which e arises naturally.
- Note: "Euler" is pronounced "Oiler", a very famous Swiss mathematician from the 18th century.

- Some students may notice intuitively that in order to be tangent to the line $y = x + 1$, the exponential function must have “slope” equal to 1 at $x = 0$. The base e is the only base that yields an exponential function with this property.

Group Work

Questions 5-7

- Have students work in groups on questions 5-7. As you direct them to continue working in their groups, emphasize what they have shown so far: the line $y = t + 1$ is tangent to the curve $y = e^t$ at the point $(0,1)$. Tell them that the “easy way” to do question 5 is to use this fact and scale the input, as directed in the paragraph before question 5. The given function $L(t)$ is the result of scaling the input of the function $f(t) = t + 1$ by 0.034. Therefore you can get a formula for $P(t)$ by scaling the input of $g(t) = e^t$ by the same constant.
- To encourage the method of scaling inputs, ask a question like, “How can you scale the input of $f(t) = t + 1$ to obtain $L(t)$?”
- Circulate and check to make sure that students are scaling the input to find the formula for $P(t)$, and not using trial and error as they did in questions 2-4.
- Even though it was reviewed in the preview assignment, it may be necessary to remind students of the power rule for exponents: $b^{uv} = \left(b^u\right)^v$.

Wrap-up/transition

(6 minutes)

Wrap-up

- Summarize the following key points of the activity:
  - The tangent line makes a good approximation near the point where it is tangent.
  - The line $y = x + 1$ is tangent to the curve $y = e^x$ at $(0,1)$.
  - The line $y = kx + 1$ is tangent to the curve $y = e^{kx}$ at $(0,1)$.
- We found two different formulas for $P(t)$: In terms of e, $P(t) = e^{0.34t}$, and as an ordinary exponential, $P(t) = (1.0346)^t$. The first form is

convenient because it shows how P is related to the linear approximation $L(t) = 0.034t + 1$.

- Linear functions (like L) have a constant rate of change per unit increase in input (slope), but exponential functions (like P) have a constant percentage increase. The example in the activity shows that these two ways of increasing are pretty close for small values of t, but become very different as t gets large.
- *Optional:* If students ask why the numbers 0.34 and 1.0346 are so similar (or if they ask why the base isn't 1.034 instead), here is one explanation: Increasing exponential functions "bend up", so they always need to be above their tangent lines. The function $(1.034)^t$ would intersect the function $0.034t + 1$ at two points ($t = 0$ and $t = 1$), so its graph would fall below the line on the interval $[0,1]$. Therefore, the base has to be a little bigger (1.0346 instead of 1.034) to make the line tangent to the curve. (Aside: Tangent lines make good approximations on both sides of the point of tangency, but we didn't look at points where $t < 0$ in this lesson.)
- Have students refer back to the **Objectives for the lesson** and check the ones they recognize from the activity. Alternatively, they may check the objectives throughout the lesson.

Transition

- "In this activity, we saw one reason why the constant e is important. The next activity will show us another situation where e occurs naturally."

Suggested assessment, assignments, and reflections

- Give Practice Assignment 24.A.
- Give the Preview Assignments, if any, for the lesson activities you plan to complete in the next class meeting.