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About the Charles A. Dana Center at The University of Texas at Austin

The Dana Center develops and scales math and science education innovations to support educators, administrators, and policy makers in creating seamless transitions throughout the K–14 system for all students, especially those who have historically been underserved.

We work with our nation's education systems to ensure that every student leaves school prepared for success in postsecondary education and the contemporary workplace—and for active participation in our modern democracy. We are committed to ensuring that the accident of where a student attends school does not limit the academic opportunities he or she can pursue. Thus, we advocate for high academic standards, and we collaborate with local partners to build the capacity of education systems to ensure that all students can master the content described in these standards.

Our portfolio of initiatives, grounded in research and two decades of experience, centers on mathematics and science education from prekindergarten through the early years of college. We focus in particular on strategies for improving student engagement, motivation, persistence, and achievement.

We help educators and education organizations adapt promising research to meet their local needs and develop innovative resources and systems that we implement through multiple channels, from the highly local and personal to the regional and national. We provide long-term technical assistance, collaborate with partners at all levels of the education system, and advise community colleges and states.

We have significant experience and expertise in the following:

- Developing and implementing standards and building the capacity of schools, districts, and systems
- Supporting education leadership, instructional coaching, and teaching
- Designing and developing instructional materials, assessments, curricula, and programs for bridging critical transitions
- Convening networks focused on policy, research, and practice

The Center was founded in 1991 at The University of Texas at Austin. Our staff members have expertise in leadership, literacy, research, program evaluation, mathematics and science education, policy and systemic reform, and services to high-need populations. We have worked with states and education systems throughout Texas and across the country. For more information about our programs and resources, see our homepage at www.utdanacenter.org.

About the Dana Center Mathematics Pathways

The Dana Center Mathematics Pathways (DCMP) is a systemic approach to improving student success and completion through implementation of processes, strategies, and structures based on four fundamental principles:

1. Multiple pathways with relevant and challenging mathematics content aligned to specific fields of study
2. Acceleration that allows students to complete a college-level math course more quickly than in the traditional developmental math sequence
3. Intentional use of strategies to help students develop skills as learners
4. Curriculum design and pedagogy based on proven practice

The Dana Center has developed curricular materials for three accelerated pathways—*Statistical Reasoning*, *Quantitative Reasoning*, and *Reasoning with Functions I* and *Reasoning with Functions II* (a two-course preparation for Calculus). The pathways are designed for students who have completed arithmetic or who are placed at a beginning algebra level. All three pathways have a common starting point—a developmental math course that helps students develop foundational skills and conceptual understanding in the context of college-level course material.

In the first term, we recommend that students also enroll in a learning frameworks course to help them acquire the strategies—and tenacity—necessary to succeed in college. These strategies include setting academic and career goals that will help them select the appropriate mathematics pathway.

In addition to the curricular materials, the Dana Center has developed tools and services to support project implementation. These tools and services include an implementation guide, data templates and planning tools for colleges, and training materials for faculty and staff.

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The development of the Dana Center Mathematics Pathways curricular materials began with the formation of the **DCMP Curricular Design Team**, who set the design standards for how the curricular materials for individual DCMP courses would be designed. The team members are:

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The Dana Center then convened faculty from each of the DCMP codevelopment partner institutions to provide input on key usability features of the instructor supports in curricular materials and pertinent professional development needs. Special emphasis was placed on faculty who need the most support, such as new faculty and adjunct faculty. The **Usability Advisory Group** members are:

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3.B	3.B	Spheres and Rates of Change *Use the relationship between volume and radius of a sphere to determine the relationship between their rates of change* *Compute rates of change*	25	21	36	3.B
3.C	-	Forming Effective Study Groups *Describe how to form and conduct an effective study group* *Identify key characteristics of effective study groups*	29	23	41	-
		Lesson 4: Equations of Circles				
4.A	4.A	Distance on the Line *Compute the distance between two numbers on the number line as represented by an absolute value* *Solve equations and inequalities involving absolute values* *Interpret intervals using absolute value notation*	31	25	45	4.A

Lesson	Preview Assignment	Lesson Title and Description	In-Class Activities with Answers	In-Class Activities (Student)	Lesson Planning Suggestions	Practice Assignment
4.B	4.B	Circles *Given a point,* P *and a positive distance,* r*, determine an equation whose graph is a circle centered at* P *and with radius* r *Given two points in the plane, determine the equation of the smallest circle containing both points* *Given two points in the plane, determine the center and radius of the smallest circle containing both points*	36	29	50	4.B
4.C	4.C	Equations of Circles *Determine the center and radius of the graph of a quadratic equation when the graph is a circle* *Sketch the graph of a quadratic equation*	39	31	54	4.C
		Lesson 5: Similar Triangles, Circular Motion, and Measuring Angles				
5.A	5.A	Moving Shadows *Use similar triangles to model static relationships between quantities* *Use similar triangles to model dynamic relationships between quantities*	43	33	60	5.A
5.B	5.B	Home Improvement *Set up and solve geometric optimization problems* *Maximize the area of a rectangle inscribed inside an equilateral triangle*	47	37	65	5.B
5.C	5.C	You Spin Me Round *Calculate the speed of an object in uniform circular motion* *Use proportional reasoning to find arc lengths and areas of circular sectors* *Determine the radian measure of the central angle of a given circular sector*	52	39	71	5.C

Lesson	Preview Assignment	Lesson Title and Description	In-Class Activities with Answers	In-Class Activities (Student)	Lesson Planning Suggestions	Practice Assignment
15.C	15.C	Choosing the Quadrant *Solve and Equation involving sine or cosine for an unknown angle in a specified quadrant or interval* *Find multiple solutions for a trigonometric equation*	204	161	222	15.C
15.D	15.D	Solving Trigonometric Equations *Find all of the solutions to an equation involving sine or cosine within a specified interval*	209	165	226	15.D
	Lesson 16: The Pythagorean Identity and Polar Curves					
16.A	16.A	The Pythagorean Identity *Prove identities using the Pythagorean identity* *Use the Pythagorean identity to find values of sine and cosine*	213	169	231	16.A
16.B	16.B	Is My Answer Right? *Use the Pythagorean identity to rewrite trigonometric expressions in equivalent forms*	216	171	236	16.B
16.C	16.C	Polar Graphs *Plot Points in Polar Coordinates* *Use Technology to produce graphs of polar curves*	220	173	241	16.C
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Student Resources

Curriculum Overview

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About *Reasoning with Functions II*

Reasoning With Functions II is designed for students who have completed *Reasoning With Functions I,* or college algebra, and plan on taking calculus courses or pursuing science, technology, engineering, or mathematics coursework that requires a thorough knowledge of functions and algebraic reasoning.

Course structure and contact hours

Reasoning With Functions II aligns with Math 2412 *Pre-Calculus Math* in the Texas Academic Course Guide Manual. It provides students the opportunity to use multiple representations and explicit covariational reasoning to investigate and explore quantities, their relationships, and how these relationships change. Students use their knowledge of functions to model and solve problems involving algebraic and transcendental functions and equations. Students also develop the algebraic tools necessary to model and solve problems using trigonometric functions and their properties.

Active and collaborative learning form the basis for each lesson, while independent learning and strong study habits are fostered through out-of-class assignments. The curriculum adheres to the DCMP Curriculum Design Standards and presents students with meaningful problems that arise from a variety of science, technology, engineering, or mathematical contexts. After completing this course, students will be prepared to take first-semester Calculus.

Reasoning With Functions II is designed to be taught as a one-semester course with four student contact hours per week or in a quarter system with an equivalent number of contact hours.

Structure of the curriculum

The curriculum is designed in 25-minute lessons, which can be pieced together to conform to any class length. These short bursts of active learning, combined with class discussion and summary, produce increased memory retention.[1] The instructor lesson plans contain facilitating questions to guide class discussions or help struggling students, suggestions for classroom pedagogy (individual work, small group experiences, think-pair-share, class discussion, or direct instruction), language and literacy support, possible student misconceptions, and explicit connections from the day's learning objectives to future course work in a STEM discipline.

Some lessons will suggest alternative pathways through the content, and instructors should feel welcome to modify their own approach to the lesson based on their personal experience and their understanding of their students.

When students are working independently outside of class, they will be offered a variety of problems that range from easy to more challenging. By having access to hints, answers, and explanations, students will receive immediate feedback on their understanding and skill mastery. This portion of the student's learning is facilitated through the use of Pearson's MyMathLab online learning platform.

A second component to these out-of-class assignments is a unique feature of the Dana Center's Dana Center Mathematics Pathways (DCMP). Students will prepare for upcoming lessons by completing Preview Assignments. In these assignments, students perform tasks such as learning new terminology, learning and practicing a skill, or starting to immerse themselves in the scenarios or problem situations that will be central to the next lesson. Before every lesson, students will read and perform a self-assessment on a set of prerequisite knowledge.

Structure of the Suggested Instructor Notes for the lessons

The main features of the **Suggested Instructor Notes** for the lessons are:

- **Overview and student objectives** – includes the constructive perseverance level (see below) of the lesson, the learning outcomes and objectives, and goals addressed.
- **Suggested resources and preparation –** includes technology needs, physical materials, and preparation needed for activities.
- **Prerequisite assumptions –** lists the skills that students need to be prepared for

[1] Sources: Buzan, T. (1989). *Master your memory* (Birmingham: Typersetters); Buzan, T. (1989). *Use your head* (London: BBC Books); Sousa, D. (2011). *How the brain learns, 4th ed.* (Thousand Oaks, CA: Corwin); Gazzaniga, M., Ivry, R. B., & Mangun, G. R. (2002). *Cognitive neuroscience: The biology of the mind, 2nd ed.* (New York: W.W. Norton); Stephane, M., Ince, N., Kuskowski, M., Leuthold, A., Tewfik, A., Nelson, K., McClannahan, K., Fletcher, C., & Tadipatri, V. (2010). Neural oscillations associated with the primary and recency effects of verbal working memory. *Neuroscience Letters*, 473, 172–177.); Thomas, E. (1972). The variation of memory with time for information appearing during a lecture. *Studies in Adult Education,* 57–62.

the lesson. The same list is given to the students in the preview assignment. Students are asked to rate their confidence level on each skill. If they struggle with transference of these skills into the new context of the lessons, the instructor can refer back to the preview questions to help students recognize that they have done similar problems.

- **Making connections –** details the main concepts that are extensions of earlier work in the course as well as connections forward in this course and later courses. This section also describes how the lesson is meaningfully connected with future coursework in a STEM discipline
- **Background context –** includes the main points of any informational pieces that were given to students in preview assignments. This time saving feature means the instructor does not have to look through the preview assignment to determine what students have done in order to prepare for the current lesson.
- **Suggested instructional plan –** includes excerpts of **Student Pages** and the following:
    - Frame the lesson – suggestions to elicit prior student knowledge, focus discussion, or motivate the current lesson.
    - Lesson activities – detailed suggestions for probing questions for students or groups, guiding questions for class discussions, problem-specific information, and other pedagogical approaches to the lesson. An approximate timeline for lesson activities is also suggested.
    - Wrap-up for the day or transition to the next activity.

The **Suggested Instructor Notes** do not summarize all ideas of the lesson; rather, they are intended to facilitate the inclusion of broader ideas. Instead of having the instructor inform them of the connections, the goal is to have students actively engaged in making those connections. This is a challenging skill that will be developed throughout the course. Early discussions are likely to be slow-starting and require a great deal of prompting. Instructors can build on what students say and can model how to express these abstract concepts. The facilitation prompts provide instructors with ideas on how to promote student discussion. As the explicit connections emerge, the instructor should record the ideas on the board and, especially early in the course, make sure students record the ideas in their notes.

- **Suggested assessment, assignments, and reflections –** includes references to the practice assignments that accompany the lessons. Occasionally, additional assessments, projects, or reflections are suggested. In addition, instructors are reminded to assign any preview assignments for upcoming lessons.
- Instructor version of student pages – includes answers and/or sample answers where appropriate. Additional space is provided for the instructor to add notes or to incorporate facilitation tips and guiding questions from the **Suggested Instructor Notes**.

Constructive perseverance level

The levels of constructive perseverance are a way to help instructors think about scaffolding productive struggle through the course. The levels should be viewed broadly as a continuum rather than as distinct, well-defined categories. In general, the level increases through the course, but this does not mean that every lesson later in the course will be a Level 3. The level is based both on the development of students and the demands of the content. Some content requires greater structure and more direct instruction. The levels of constructive perseverance are as follows:

- **Level 1:** The problem is broken into sub-questions that help develop strategies. Students reflect on and discuss questions briefly and then are brought back together to discuss with the full class. This process moves back and forth between individual or small-group discussion and class discussion in short intervals.

  Goal of the instructor: Develop the culture of discussion, establish norms of listening, and model the language used to discuss quantitative concepts. In addition, emphasize to students that struggling indicates learning. If struggle is not taking place, students are
  not being challenged and are not gaining new knowledge and skills.

- **Level 2:** The problem is broken into sub-questions that give students some direction but do not explicitly define or limit strategies and approaches. Students work in groups on multiple steps for longer periods, and the instructor facilitates individual groups, as needed. The instructor brings the class together at strategic points at which important connections need to be made explicit or when breakdowns of understanding are likely to occur.

  Goal of the instructor: Support students in working more independently and evaluating their own work so they feel confident about moving through multiple questions without constant reinforcement from the instructor.

- **Level 3:** The problem is not broken into steps or is broken into very few steps. Students are expected to identify strategies for themselves. Groups work independently on the problem with facilitation by the instructor, as necessary. Groups report on results, and class discussion focuses on reflecting on the problem as a whole.

  Goal of the instructor: Support students in persisting with challenging problems, including trying multiple strategies before asking for help.

Table of contents information

The table of contents contains the following information:

- Lesson number

- Preview assignment, if any – Preview assignments contain activities to help familiarize students with a concept or problem situation for the upcoming lesson as well as problems designed to assess student readiness for the prerequisite assumptions of the lesson. Students are instructed to seek help before the next class meeting if they are unable to successfully complete these problems.
- Lesson title and brief description
- In-Class Activities (Instructor) reference
- In-Class Activities (Student) reference
- Practice assignment, if any – Practice assignments consist of problems designed to assess student understanding of the concepts addressed in the lesson.

The role of the preview and practice assignments

One of the most important aspects of the *Reasoning with Functions II* curriculum is the role and design of the homework assignments. These assignments differ from traditional homework in several ways:

- The preview assignments are designed to prepare students for the next lesson. This preparation is done explicitly. These assignments allow students to familiarize themselves with a context or scenario that will be studied in the next lesson and even begin some of the more straight-forward calculations. This allows more class time to be spent discussing more subtle or complex problems. Additionally, students are given a prerequisite set of skills for the next lesson and asked to rate themselves. Each of these prerequisite skills is used in the assignment.
- The preview assignments occasionally contain information or questions that are directly used in the next lesson. These will generally be referenced in the **Suggested Instructor Notes** under **Background context.**
- The practice assignments review previous material and allow students to practice and develop skills from the current lesson.
- The design of the practice assignments is based on the same principle of constructive perseverance as the rest of the curriculum. Ideally, each assignment should offer entry- level questions that all students should be able to complete successfully and also more challenging questions. One goal of the entire curriculum is that students will increasingly engage in productive struggle. The expectation is not that every student should be able to answer every question correctly, but that every student should make a valid attempt on each question. Therefore, there are questions in the assignments, especially later in the course, that many students may not answer correctly. These challenging questions raise issues about grading practices, as discussed in the following paragraphs.
- In some cases, the assignments include actual instructional materials. It is expected that students will read these materials, as they are usually not presented directly in class. This information distinguishes the material from traditional

textbooks in which the text is often assigned by instructors, but often only used by students as a reference.

- The assignments purposefully refer students back to previous lessons. This is done to support students in making connections across the course, encourage students to review previous material, and support good organizational habits.

Strategies for supporting the assignments

The central role and unique design of the assignments in the curriculum require instructors to develop strategies and procedures for supporting students to use the assignments appropriately. The following are some areas instructors should consider and some suggestions for strategies.

Motivating students to complete the assignments – The design of the assignments supports motivation as students come to realize that much of the material is actually useful to them in class. Instructors can support student motivation by doing the following:

- Discuss the role of the assignments with students.
- Set and maintain an expectation that students should be able to use the prerequisite skills for a lesson. Students may take this lightly at first, so it is important that instructors do not use class time to review these skills but make it clear that students are responsible for being prepared. Keep in mind the following:
    - Students may have prepared but may not recognize that what they are being asked to do in class is the same skill they used in the preview questions. Be prepared to refer back to specific questions to help them make this connection.
    - If a student is truly unprepared, do not reprimand him or her in front of the class. Privately explain the expectation for preparation to the student and invite him or her to meet with you outside of class to review the material. If you do meet the student outside of class, take the opportunity to talk about the importance of preparation and inquire about how the student does the self-assessment. Help the student develop strategies for using this tool more effectively.
- Occasionally students are directed to bring a separate copy of their work to class for discussion. In this case, have some way for students who do the work to receive credit. The group work model allows all students to participate in the lesson even if they did not do the preview questions. This is important, but students who come prepared should feel that their work is valued. You can give students a quick completion grade by walking around the room and seeing who has their printed work complete while students are working with their groups.
- Notify students at the end of the first, second, and third weeks if they have failed to complete any of their work. This notification can be done by email or by handing out notes in class. It is important for students to know that the instructor is aware of their individual work. Always include an offer of help and expressions of

support in these notices. For example, “If there is something preventing you from completing your work, please come to see me. I want to help you be successful in this course.” Keep in mind that there are many reasons that students fail to complete out-of-class work.

- Written communication is an important component of the course (see the **Language and literacy skills** section in this Overview). Attention to correct mathematical notation and commenting on the reasonableness of one’s answers are expected. Often, students are specifically prompted to write answers in complete sentences. Early in the course, take the time to grade and/or provide specific feedback to students on these communication skills.

Grading assignments – Since the assignments are designed to challenge students and promote constructive perseverance, grading only on correct answers may not always be appropriate and may discourage students. On the other hand, grading on completion has drawbacks as well. Effective grading strategies have to be individualized depending on the grading time instructors have, the length of classes, and the student population. Some ideas follow.

- Use a scoring method that gives points for both completion and correctness.
- Grade on correctness but occasionally allow students to turn in written explanations for problems they missed and earn back points. This work can be managed by limiting the opportunity to one or two problems each week or to certain assignments.

Many students struggle with organization. Instructors should provide some sort of structure to support students. Strategies include the following:

- Explain to students why it is important to organize their materials. Give specific examples of the ways in which they will use the materials in this course.
- Require that students keep materials in a three-ring binder.
    - **High structure:** Give students guidelines on how to order and label materials.
    - **Moderate structure:** Give students guidelines but also give them the option to create their own method of organization.
- Any structure that is required should be graded in some way in order to encourage students to complete it. Checks should be done in the first few weeks of the course to establish a routine.
    - Check in class on a regular basis: Tell students to find a specific document within a specified amount of time (e.g., 2 minutes). Students get a grade for showing the instructor the document.
    - Start with a quick check for having the system (e.g., binder, folder) set up. Then occasionally have students turn their materials in and do a spot-check for certain documents.
    - Give timed quizzes in which students are referred to certain documents and must respond to some quick question about the materials.

Resource materials for students

The student resource packet is designed to be the starting point for course reference materials. Some of the resources are directly lifted from an assignment because they contain material that may be useful to students later in the course. Other resources were created as supplementary material. Encourage students to keep a section of their class binder dedicated to the provided resources as well as any additional resources they may collect.

Language and literacy skills

All STEM disciplines desire students that are able to communicate mathematical and quantitative ideas. Students who excel at traditional algebraic manipulation and mathematical computations often struggle with interpreting and explaining mathematical information in conjunction with language. Students that have completed *Reasoning With Functions I* have already developed some communication skills and should be comfortable with using appropriate language, symbolism, data, and graphs to communicate quantitative results. *Reasoning with Functions II* will build upon this foundation and expect students to incorporate the correct use of mathematical notation and language into their written responses.
The learning outcomes of the course include the following:

- Communicate effectively about function processes using function notation.
- Describe the behavior of functions on entire intervals.
- Describe dynamic scenarios orally and in written form using appropriate mathematical language.
- Communicate their conclusions both orally and in written form and support their conclusions by providing appropriate mathematical justifications.

Students who completed the earlier course *Reasoning With Functions I* have substantial practice in mathematical literacy. Students should be able to write two to three paragraphs that make appropriate use of quantitative information. *Reasoning with Functions II* will build on this foundation and continue to expect students to explain their reasoning and justify their conclusions using complete sentences and other excellent written communication skills. The Resource **Writing Principles** supports this work.

Content outline of the curriculum

The following outline gives an overview of the curriculum.

Unit	Lessons	Concepts & Skills
Geometry Approximately 1 week	1A – 1B	Approximating area under a graph; Optimizing area and perimeter of rectangular regions.
	2A – 2C	Determining area of triangles and understanding the effects of changing dimensions of a triangle on area; Pythagorean Theorem, distance between two points, and estimating length of a curve.
	3A – 3B	Computing rate of change, determining average rate of change of the circumference and area of a circle, and determining average rate of change of the volume of a disk.
Dynamic Situations and Circular Functions Approximately 3 weeks	4A – 4C	Using absolute value to compute distance between two numbers, solving equalities and inequalities involving absolute value, and expressing intervals using absolute value notation; Determining equations of circles, and identifying equations that represent circles.

	5A – 5C	Similar triangles and similar triangles in dynamic situations; Maximizing area of inscribed rectangles; Uniform circular motion; Arc length, areas of circular sectors, and radian measure.
	6A – 6D	Calculating surface area and volume of a cylinder; Surface areas of cones and frustums, and modeling changing dimensions in a cone; Estimating instantaneous rates of change of dimensions in a cone, and maximizing volume of a cone.
	7A – 7C	Understanding sinusoidal graphs (period, amplitude, midline), and modeling physical situations.
	8A – 8D	The unit circle, interpreting sinusoidal functions in the context of a model, special angles, and analyzing sinusoidal graphs.
Assessment		

Sinusoidal Models, Transformations, and Rate of Change Approximately 2 weeks	9A – 9C	Modeling physical situations using non-unit circles; Describing how changes in angle and radius affect locations of points; Damped harmonic motion.
	10A – 10C	Modeling oscillations, understanding frequency, and recognizing horizontal shifts.
	11A – 11C	Maximum and minimum values of sinusoidal models, and recognizing vertical shifts.
	12A – 12D	Average rate of change of sinusoidal functions.

Right Triangle Trigonometry and Inverse Functions Approximately 1.5 weeks	13A – 13D	Trigonometric ratios in triangles; Solving right triangles in various scenarios.
	14A – 14C	Restricted domain of trigonometric functions; Inverse trigonometric functions, domain and range of inverse trigonometric functions, and average rate of change of inverse trigonometric functions.
Assessment		

Trigonometric Identities Approximately 2.5 weeks	15A – 15D	Finding multiple solutions to trigonometric equations; Modeling physical situations using inverse trigonometric functions.
	16A – 16D	The Pythagorean Identity; Proving identities; Polar coordinates and polar curves.
	17A – 17D	Sum and Difference Identities for Sine and Cosine; Sum-to-Product Identities.
	18A – 18D	Double Angle Formulas for Sine and Cosine; Half-Angle Formulas.
Assessment		

Other Trigonometric Functions Approximately 3	19A – 19E	The Law of Sines; The Law of Cosines.
	20A – 20E	The tangent function, its graph, and its inverse; The secant function and its graph; Identities involving tangent and secant.
	21A – 21D	The cotangent function, its graph, and its inverse; The cosecant function, its graph, and its inverse; Identities involving the cosecant function.
	22A – 22D	Models involving both linear and cyclic behavior.
Assessment		
Final Exam		

DCMP Curriculum design standards

The Dana Center Mathematics Pathways (DCMP) is made up of individual courses that form *pathways* for students to and through college-level mathematics. The concept of the pathway as a yearlong experience is critical to the DCMP because these courses are designed to articulate with each other to provide students with the experience of learning mathematics and/or statistics through coherent, consistent practices and structures.

The design standards outlined in this section set the guidelines for how the curricular materials for individual DCMP courses are designed to support that coherent experience for students.

Note: The numbering in the description of the design standards does not indicate level of importance.

Standard I: Structure and Organization of Curricular Materials

The DCMP is organized around big mathematical and statistical ideas and concepts as opposed to skills and topics.

Standard II: Active Learning

The DCMP is designed to actively involve you in doing mathematics and statistics, analyzing data, constructing hypotheses, solving problems, reflecting on your work, and learning and making connections.

Class activities provide regular opportunities for you to actively engage in discussions and tasks using a variety of different instructional strategies (e.g., small groups, class discussions, interactive lectures).

Standard III: Constructive Perseverance

The DCMP supports students in developing the tenacity, persistence, and perseverance necessary for learning mathematics.

Standard IV: Problem Solving

The DCMP supports you in developing problem-solving skills and in applying previously learned skills to solve nonroutine and unfamiliar problems.

Standard V: Context and Interdisciplinary Connections

The DCMP presents mathematics and statistics in context and connects mathematics and statistics to various disciplines.

Standard VI: Use of Terminology

The DCMP uses discipline-specific terminology, language constructs, and symbols to intentionally build mathematical and statistical understanding and to ensure that terminology is not an obstacle to understanding.

Standard VII: Reading and Writing

The DCMP develops your ability to communicate about and with mathematics and statistics in contextual situations appropriate to the pathway.

Standard VIII: Technology

The DCMP uses technology to facilitate active learning by enabling you to directly engage with and use mathematical concepts. Technology should support the learning objectives of the lesson. In some cases, the use of technology may be a learning objective in itself, as in learning to use a statistical package in a statistics course.

Readiness competencies

Students enrolling in *Reasoning With Functions II* should be able to do the following:

- Create and interpret mathematical models within a variety of contexts.
- Communicate effectively with function notation.
- Use multiple representations of functions to interpret and describe how two quantities change together, justify the presence of a relationship, identify constraints, distinguish between dependent and independent variables, identify domains and ranges, and draw diagrams of dynamic situations.
- Compute, describe, and interpret rates of change embedded in multiple function representations.
- Analyze linear, polynomial, exponential, power, rational, and logarithmic functions as well as their inverses and compositions.
- Demonstrate procedural fluency in applying factoring techniques to simplify expressions and locate roots, interpreting and evaluating expressions involving variables, solving mathematical equations, and reading and applying formulas.
- Take charge of their own learning through good classroom habits, time management, and persistence. Participate in the classroom community through written and oral communication.

Learning goals

The following five learning goals apply to all DCMP mathematics courses, with the complexity of problem-solving skills and use of strategies increasing as students advance through the pathways.

For each course, we define the ways that the learning goals are applied and the expectations for mastery. The bullets below each of the five learning goals specify the ways each learning goal is applied in the *Reasoning With Functions II* course.

Each DCMP course is designed so that students meet the goals across the courses in a given pathway. Within a course, the learning goals are addressed across the course's content-based learning outcomes.

Communication Goal: Students will be able to interpret and communicate quantitative information and mathematical and statistical concepts using language appropriate to the context and intended audience.

In the *Reasoning With Functions II* course, students will...

- communicate effectively about function processes using function notation.
- describe the behavior of functions on entire intervals.
- describe dynamic scenarios orally and in written form using appropriate mathematical language.
- communicate their conclusions both orally and in written form and support their conclusions by providing appropriate mathematical justifications.

Problem Solving Goal: Students will be able to make sense of problems, develop strategies to find solutions, and persevere in solving them.

In the *Reasoning With Functions II* course, students will...

- identify a variety of strategies to solve a problem, persist in applying a strategy, and reflect on the outcome of that strategy.
- solve complex problems in a variety of contexts related to science, technology, engineering, or mathematics.

Reasoning Goal: Students will be able to reason, model, and draw conclusions or make decisions with mathematical, statistical, and quantitative information.

In the *Reasoning With Functions II* course, students will...

- apply covariational reasoning skills in various contexts and representations and draw appropriate conclusions.
- apply dynamic reasoning to create appropriate models and use these models to make decisions.
- create mathematical models in a variety of meaningful mathematical applications and use these models to make decisions.

Evaluation Goal: Students will be able to critique and evaluate quantitative arguments that utilize mathematical, statistical, and quantitative information.

In the *Reasoning With Functions II* course, students will...

- identify constraints and limitations for mathematical models in a variety of contexts and representations.
- critically reflect on the reasonableness of their solutions.

Technology Goal: Students will be able to use appropriate technology in a given context.

In the *Reasoning With Functions II* course, students will...

- use technology effectively and appropriately to analyze multiple representations of functions.

Content learning outcomes

The final content and sequencing of *Reasoning With Functions I* comes from examining current research in mathematics education together with recommendations from outreach work to mathematics' partner disciplines. In addition to specific content required for success in calculus, four overarching principles drive the content of the STEM Prep pathway:

- Deep understanding of the function process: A strong conceptual understanding of the process view of function (contrasted with an action view) will give students a critical mathematical foundation to support their future coursework within STEM fields. By stressing the process view of a function, the *Reasoning with Functions* curriculum will prepare students to analyze function outputs on entire intervals of inputs, help students to reason about inverting functions by reversing a process, and make stronger connections between the graph of a function and its relationship to generalized inputs and outputs.
- Covariational reasoning: The ability to simultaneously analyze two quantities, how they change, and how they co-vary enables students to better understand the unique and dynamic problem situations which populate the study of calculus and other STEM disciplines. The *Reasoning with Functions* curriculum will give students many opportunities to explore dynamic function relationships and allow them to more easily conceptualize the notions of an average rate of change and transition between an average rate of change and an instantaneous rate of change.
- Communication with functions and function notation: Students will communicate orally and in writing as they analyze function behavior from multiple representations. The *Reasoning with Functions* curriculum will engage students at the notational level by directly examining the need for function notation, and by requiring students to interpret to and from function notation.

- Meaningful approaches to algebraic reasoning: Students will engage with the curriculum as they develop their algebra and problem solving skills within authentic STEM contexts and models. Students will create, explore, and interpret mathematical models and use algebra as a way of extracting additional information from a model or mathematical problem.

The learning outcomes for *Reasoning with Functions I* are organized around three topics:

- Foundations of Functions
- Analysis of Functions
- Algebraic Reasoning

Geometric Reasoning

Outcome: Students will apply geometric reasoning to model and solve problems involving length, area, and volume.

Students will be able to:

G.1 Use geometric formulas for length, area, and volume of common shapes.

For example: Apply the Pythagorean Theorem and determine the distance between two points in the plane. Compute circumference and area of circles. Compute perimeter and area of triangles and rectangles. Calculate the volume of spheres, rectangular solids, cones, and cylinders.

G.2 Use proportional reasoning to describe and identify geometric quantities.

For example: Determine the arc length of a sector. Find missing lengths or angles in similar triangles.

Trigonometry

Outcome: Students will model and solve meaningful problems using trigonometric functions and their properties.

Students will be able to:

T.1 Use the basic trigonometric functions to model and solve meaningful problems.

For example: Convert between degrees and radians. Interpret sine and cosine as coordinates on a (unit) circle. Understand definitions of tangent, cotangent, secant, and cosecant. Apply right triangle trigonometry and recognize when a trigonometric function appropriately represents the relationship between two quantitative variables.

T.2 Prove and use trigonometric identities.

For example: Use the Pythagorean identity (and its variations), double and half angle identities, and angle addition and subtraction formulas to convert and simplify trigonometric expressions.

T.3 Identify important properties of trigonometric functions.

For example: Identify amplitude, period, frequency, phase shift (domain shift), and vertical and horizontal shifts and stretches.

T.4 Solve for missing lengths or angles of oblique triangles.

For example: Apply the Law of Sines and Cosines.

T.5 Use and describe inverse trigonometric functions.

For example: Use a calculator and principal angle to evaluate inverse trigonometric functions and solve equations using properties of inverse trigonometric functions.

Functions

Outcome: Students will use and apply knowledge of functions to model and solve meaningful problems.

Students will be able to:

F.1 Model and solve meaningful problems involving algebraic and transcendental functions and equations.

For example: Create models using algebraic and graphical problem solving techniques and properties of functions.

Suggestions for Prep Week

Introduction

The design of this course gives you flexibility in choosing an appropriate pace for your students. The activities each take about 25 minutes and can be grouped to fit any time structure.

As you think about teaching this course, decide how many lessons you will aim to cover each day. Most lessons are presented in three to five parts.

When preparing materials for students, keep in mind that in general, each lesson's multiple parts do not have to be printed on separate pages. For example, if you are distributing hard copies of student pages for the first day and are covering Lesson 1, Parts A and B, all these pages can be produced double-sided. Printing double-sided pages will help the class keep a cohesive flow, instead of giving the impression of separate sections.

First week sequencing

The first few days of the course are designed to accomplish multiple goals:

- Students will do mathematics.
- Students will interact with one another and the instructor.
- Students will begin to develop an understanding of the structure of the course, which is designed around:
    - Active learning
    - Constructive perseverance
    - Learning community

For 50-minute classes

Day 1:

Lesson 1, Part A ("Approximating Area")

Pass out and discuss syllabus

Introduce students to online platform

Assign Practice Assignment 1.A

Assign Preview Assignment 1.B (to be completed before the next class meeting)

Day 2:

Lesson 1, Part B ("Optimization and Rectangles") and **Lesson 1, Part C** ("Our Learning Community" — student success lesson)

Assign Practice Assignment 1.B

Assign Preview Assignments 2.A and 2.B (to be completed before the next class meeting)

Day 3:

Lesson 2 Part A (“Geometry of Triangles: Area”) and **Lesson 2, Part B** (“Right Triangles and Rates”)

Assign Practice Assignments 2.A and 2.B

Assign Preview Assignment 2.C and 3.A (to be completed before the next class meeting)

Day 4:

Lesson 2, Part C (“Distance and Arc Length”) and **Lesson 3, Part A** (“Circles and Rates of Change”)

Assign Practice Assignments 2.C and 3.A Assign Preview Assignment 3.B

For 75- to 100-minute classes

Day 1:

Lesson 1, Part A (“Approximating Area”) and **Lesson 1, Part B** (“Optimization and Rectangles”)

Introduce students to online platform. Explain the importance of completing preview assignments outside of class and, if possible, ask students to complete Preview Assignment 1.B as an introduction to the online platform.

Pass out and discuss syllabus Introduce students to online platform

Assign Practice Assignments 1.A and 1.B

Assign Preview Assignments 2.A and 2.B (to be completed before the next class meeting)

Day 2:

Lesson 1, Part C (“Our Learning Community” — student success lesson), **Lesson 2, Part A** (“Geometry of Triangles: Area”), and **Lesson 2, Part B** (“Right Triangles and Rates”)

Assign Practice Assignments 2.A and 2.B.

Assign Preview Assignments for 2.C, 3.A, and 3.B (to be completed before the next class meeting).

Day 3:

Lesson 2, Part C (“Distance and Arc Length”), **Lesson 3, Part A** (“Circles and Rates of Change”) and **Lesson 3, Part B** (“Spheres and Rates of Change”)

Assign Practice Assignments 2.C, 3.A, and 3.B.

Assign Preview Assignment for 4.A and 4.B (to be completed before the next class meeting).

Syllabus ideas

1. Consider having students keep a class binder.

   Binders may include sections for resources, student work, and assessment items (that have been graded and returned).

   Recommend to students that to show their work, they insert loose-leaf paper in their binders between their course student pages. (If they prefer to keep their work in a spiral notebook, then they need to develop the habit of consistently labeling the work paper with the lesson number and letter.)

   The student lesson pages are intentionally designed without work space. Many students feel constrained by work spaces of specified lengths, so working on their own paper gives them the freedom to show as much work as necessary and have sufficient room to provide justifications, regardless of the size of their handwriting.

   Students coming from the Dana Center Mathematics Pathways (DCMP) *Reasoning With Functions I* course may be familiar with this format, but other students may not have a reference point for what it looks like to have well-organized course materials.

   To help students develop these skills, consider awarding some type of credit—such as a quiz grade for a couple of binder checks over the course of the semester—to students who keep their binders according to your guidelines. This reward system reinforces the message that you believe organization is beneficial, and the experience will help students reap the benefits inherent in organizing their work. Consequently, they are more likely to begin to self-regulate these behaviors.

2. Consider including the following statement in your syllabus:

   This is a mathematics course in which you will learn to use, understand, and communicate about mathematical information. The course has five goals:

   - **Communication goal:** You will interpret and communicate quantitative information and mathematical concepts using language appropriate to the context and intended audience.
   - **Problem Solving goal:** You will make sense of problems, develop strategies to find solutions, and persevere in solving them.
   - **Reasoning goal:** You will reason, model, and make decisions with mathematical and quantitative information.
   - **Evaluation goal:** You will critique and evaluate quantitative arguments that utilize mathematical and quantitative information attending to precision in all of your work.
   - **Technology goal:** You will use appropriate technology in a given context.

3. Consider including information about productive struggle and constructive perseverance (see the Curriculum Overview).

Classroom routines

Determine classroom routines you want to establish and consider which ones you wish to describe in the syllabus.

Instructors can establish a few routines to emphasize and support certain behaviors. These routines should be explained to the students and started immediately at the beginning of the term. Routines should be kept at a minimum because their primary value is in consistency. You do not want to overwhelm yourself or your students with tasks. Below are examples of routines that can support productive connections with students.

- To encourage mutual support, have students form "buddy groups" in which they exchange contact information. The group members are responsible for sharing information if anyone is absent.
- Greet students at the door as they arrive.
- Establish a practice of having a quick, personal conversation with 5 students every day. Make sure you cycle your way to all of your students (depending on the size of your class, you may go through several cycles during the semester).
- To encourage attendance, start each class by asking the students who is absent, and note those absent students on the board. This routine encourages students to be aware of one another and indicates that someone will notice if they are gone. Have students call absent classmates to offer help with missed material.
- Keep track of students you interact with during each class period. If possible, ensure that you interact with each student at least once every week.

A culture of discourse

The first few lessons are designed to introduce students to mathematics, set the stage for this course, and establish a culture of discourse that is faithful to DCMP principles.

How you facilitate the group work and class discussion in the first few class meetings sets the stage for the nature of the discourse in future classes. Be prepared with strategies that encourage students to speak up in class. For example:

- Listen to groups for observations, comments, or questions that students can share with the class.
- Think–Pair–Share is a quick and easy occasional alternative to group work. This strategy allows students to think about a question on their own for a short time, discuss with a partner, and then share with the class.
- Do not allow individuals to dominate. Call on individual students or ask for a response from a certain part of the room (this can be less stressful than calling on a student). For example, say, "Now I'd like someone in the back row to answer." Make it clear that blurting out an answer is not acceptable.

- Encourage students to respond to one another.
- Avoid standing at the front of the room when possible—if you are in the front, then you are the focus.
- Honor any serious contribution. Thank students in class and after class for their comments.

DCMP *Reasoning With Functions I* lessons revolve around problem situations that use mathematical calculations to achieve a deep understanding of the following:

- Function processes
- Covariational reasoning
- Communication with functions and function notation
- Meaningful approaches to algebraic reasoning.

It is important to note that the **calculations in isolation are not the objective**. Instead, problem situations are designed to develop the skills listed above in preparation for calculus.

In teaching each lesson, allow students to do enough of the math to make sure they understand the situation and start to recognize that mathematical tools can be useful, but don't allow them to get bogged down or become frustrated with calculations. Productive struggle revolves around thinking about situations and identifying which skills are necessary to answer questions.

These tips will help students develop a feeling of safety, which will pay off with increased student engagement and participation.